Standard Practice for Reporting Uniaxial Strength Data and Estimating Weibull Distribution Parameters for Advanced Ceramics[1]

This standard is issued under the fixed designation C 1239; the number immediately following the designation indicates the year of original adoption or, in the case of revision, the year of last revision. A number in parentheses indicates the year of last reapproval. A superscript epsilon (ϵ) indicates an editorial change since the last revision or reapproval.

1. Scope

1.1 This practice covers the evaluation and subsequent reporting of uniaxial strength data and the estimation of probability distribution parameters for advanced ceramics that fail in a brittle fashion. The failure strength of advanced ceramics is treated as a continuous random variable. Typically, a number of test specimens with well-defined geometry are failed under well-defined isothermal loading conditions. The load at which each specimen fails is recorded. The resulting failure stresses are used to obtain parameter estimates associated with the underlying population distribution. This practice is restricted to the assumption that the distribution underlying the failure strengths is the two-parameter Weibull distribution with size scaling. Furthermore, this practice is restricted to test specimens (tensile, flexural, pressurized ring, etc.) that are primarily subjected to uniaxial stress states. Section 8 outlines methods to correct for bias errors in the estimated Weibull parameters and to calculate confidence bounds on those estimates from data sets where all failures originate from a single flaw population (that is, a single failure mode). In samples where failures originate from multiple independent flaw populations (for example, competing failure modes), the methods outlined in Section 8 for bias correction and confidence bounds are not applicable.

1.2 Measurements of the strength at failure are taken for one of two reasons: either for a comparison of the relative quality of two materials, or the prediction of the probability of failure (or, alternatively, the fracture strength) for a structure of interest. This practice will permit estimates of the distribution parameters that are needed for either. In addition, this practice encourages the integration of mechanical property data and fractographic analysis.

1.3 This practice includes the following:

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1.4 The values stated in SI units are to be regarded as the standard.

2. Referenced Documents

2.1 *ASTM Standards:*

C 1145 Terminology of Advanced Ceramics[2]

C 1322 Practice for Fractography and Characterization of Fracture Origins in Advanced Ceramics[2]

D 4392 Terminology for Statistically Related Terms[3]

E 6 Terminology Relating to Methods of Mechanical Testing[4]

E 178 Practice for Dealing With Outlying Observations[5]

E 456 Terminology Relating to Quality and Statistics[5]

2.2 *Military Handbook:*

MIL-HDBK-790 Fractography and Characterization of Fracture Origins in Advanced Structural Ceramics[6]

3. Terminology

3.1 Proper use of the following terms and equations will alleviate misunderstanding in the presentation of data and in the calculation of strength distribution parameters.

3.1.1 *censored strength data*—strength measurements (that is, a sample) containing suspended observations such as that produced by multiple competing or concurrent flaw populations.

3.1.1.1 Consider a sample where fractography clearly established the existence of three concurrent flaw distributions (although this discussion is applicable to a sample with any number of concurrent flaw distributions). The three concurrent flaw distributions are referred to here as distributions *A*, *B*, and *C*. Based on fractographic analyses, each specimen strength is

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[2] *Annual Book of ASTM Standards*, Vol 15.01.

[3] *Discontinued*—see 1992 *Annual Book of ASTM Standards*, Vol 07.02.

[4] *Annual Book of ASTM Standards*, Vol 03.01.

[5] *Annual Book of ASTM Standards*, Vol 14.02.

[6] Available from Standardization Documents Order Desk, Bldg. 4 Section D, 700 Robbins Ave., Philadelphia, PA 19111-5094, Attn: NPODS.

assigned to a flaw distribution that initiated failure. In estimating parameters that characterize the strength distribution associated with flaw distribution *A*, all specimens (and not just those that failed from Type A flaws) must be incorporated in the analysis to ensure efficiency and accuracy of the resulting parameter estimates. The strength of a specimen that failed by a Type B (or Type C) flaw is treated as a *right censored* observation relative to the *A* flaw distribution. Failure due to a Type B (or Type C) flaw restricts, or censors, the information concerning Type A flaws in a specimen by suspending the test before failure occurred by a Type A flaw **(1)**.[7] The strength from the most severe Type A flaw in those specimens that failed from Type B (or Type C) flaws is higher than (and thus to the *right* of) the observed strength. However, no information is provided regarding the magnitude of that difference. Censored data analysis techniques incorporated in this practice utilize this incomplete information to provide efficient and relatively unbiased estimates of the distribution parameters.

3.2 *Definitions:*

3.2.1 *competing failure modes*—distinguishably different types of fracture initiation events that result from concurrent (competing) flaw distributions.

3.2.2 *compound flaw distributions*—any form of multiple flaw distribution that is neither pure concurrent nor pure exclusive. A simple example is where every specimen contains the flaw distribution *A*, while some fraction of the specimens also contains a second independent flaw distribution *B*.

3.2.3 *concurrent flaw distributions*—a type of multiple flaw distribution in a homogeneous material where every specimen of that material contains representative flaws from each independent flaw population. Within a given specimen, all flaw populations are then present concurrently and are competing with each other to cause failure. This term is synonymous with ″competing flaw distributions."

3.2.4 *effective gage section*—that portion of the test specimen geometry that has been included within the limits of integration (volume, area, or edge length) of the Weibull distribution function. In tensile specimens, the integration may be restricted to the uniformly stressed central gage section, or it may be extended to include transition and shank regions.

3.2.5 *estimator*—a well-defined function that is dependent on the observations in a sample. The resulting value for a given sample may be an estimate of a distribution parameter (a point estimate) associated with the underlying population. The arithmetic average of a sample is, for example, an estimator of the distribution mean.

3.2.6 *exclusive flaw distributions*—a type of multiple flaw distribution created by mixing and randomizing specimens from two or more versions of a material where each version contains a different single flaw population. Thus, each specimen contains flaws exclusively from a single distribution, but the total data set reflects more than one type of strength-controlling flaw. This term is synonymous with ″mixtures of flaw distributions."

3.2.7 *extraneous flaws*—strength-controlling flaws observed in some fraction of test specimens that cannot be present in the component being designed. An example is machining flaws in ground bend specimens that will not be present in as-sintered components of the same material.

3.2.8 *fractography*—the analysis and characterization of patterns generated on the fracture surface of a test specimen. Fractography can be used to determine the nature and location of the critical fracture origin causing catastrophic failure in an advanced ceramic test specimen or component.

3.2.9 *multiple flaw distributions*—strength controlling flaws observed by fractography where distinguishably different flaw types are identified as the failure initiation site within different specimens of the data set and where the flaw types are known or expected to originate from independent causes.

3.2.9.1 *Discussion*—An example of multiple flaw distributions would be carbon inclusions and large voids which may both have been observed as strength controlling flaws within a data set and where there is no reason to believe that the frequency or distrubution of carbon inclusions created during fabrication was in any way dependent on the frequency or distribution of voids (or vice-versa).

3.2.10 *population*—the totality of potential observations about which inferences are made.

3.2.11 *population mean*—the average of all potential measurements in a given population weighted by their relative frequencies in the population.

3.2.12 *probability density function*—the function $f(x)$ is a probability density function for the continuous random variable X if:

$$f(x) \geq 0 \tag{1}$$

and

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \tag{2}$$

The probability that the random variable X assumes a value between a and b is given by the following equation:

$$Pr(a < X < b) = \int_a^b f(x)\, dx \tag{3}$$

3.2.13 *sample*—a collection of measurements or observations taken from a specified population.

3.2.14 *skewness*—a term relating to the asymmetry of a probability density function. The distribution of failure strength for advanced ceramics is not symmetric with respect to the maximum value of the distribution function but has one tail longer than the other.

3.2.15 *statistical bias*—inherent to most estimates, this is a type of consistent numerical offset in an estimate relative to the true underlying value. The magnitude of the bias error typically decreases as the sample size increases.

3.2.16 *unbiased estimator*—an estimator that has been corrected for statistical bias error.

3.2.17 *Weibull distribution*—the continuous random variable X has a two-parameter Weibull distribution if the probability density function is given by the following equations:

[7] The boldface numbers in parentheses refer to the list of references at the end of this practice.

$$f(x) = \left(\frac{m}{\beta}\right)\left(\frac{x}{\beta}\right)^{m-1} exp\left[-\left(\frac{x}{\beta}\right)^{m}\right] \quad x>0 \tag{4}$$

$$f(x) = 0 \quad x \leq 0 \tag{5}$$

and the cumulative distribution function is given by the following equations:

$$F(x) = 1 - exp\left[-\left(\frac{x}{\beta}\right)^{m}\right] \quad x > 0 \tag{6}$$

or

$$F(x) = 0 \quad x \leq 0 \tag{7}$$

where
m = Weibull modulus (or the shape parameter) (>0), and
β = scale parameter (>0).

3.2.18 The random variable representing uniaxial tensile strength of an advanced ceramic will assume only positive values, and the distribution is asymmetrical about the mean. These characteristics rule out the use of the normal distribution (as well as others) and point to the use of the Weibull and similar skewed distributions. If the random variable representing uniaxial tensile strength of an advanced ceramic is characterized by Eq 4-7, then the probability that this advanced ceramic will fail under an applied uniaxial tensile stress σ is given by the cumulative distribution function as follows:

$$P_f = 1 - exp\left[-\left(\frac{\sigma}{\sigma_\theta}\right)^{m}\right] \sigma > 0 \tag{8}$$

$$P_f = 0 \quad \sigma \leq 0 \tag{9}$$

where:
P_f = probability of failure, and
σ_θ = Weibull characteristic strength.

Note that the Weibull characteristic strength is dependent on the uniaxial test specimen (tensile, flexural, or pressurized ring) and will change with specimen size and geometry. In addition, the Weibull characteristic strength has units of stress and should be reported using units of megapascals or gigapascals.

3.2.19 An alternative expression for the probability of failure is given by the following equation:

$$P_f = 1 - exp\left[-\int_v \left(\frac{\sigma}{\sigma_0}\right)^{m} dV\right] \sigma > 0 \tag{10}$$

$$P_f = 0 \quad \sigma \leq 0 \tag{11}$$

The integration in the exponential is performed over all tensile regions of the specimen volume if the strength-controlling flaws are randomly distributed through the volume of the material, or over all tensile regions of the specimen area if flaws are restricted to the specimen surface. The integration is sometimes carried out over an effective gage section instead of over the total volume or area. In Eq 10, σ_0 is the Weibull material scale parameter. The parameter is a material property if the two-parameter Weibull distribution properly describes the strength behavior of the material. In addition, the Weibull material scale parameter can be described as the Weibull characteristic strength of a specimen with unit volume or area loaded in uniform uniaxial tension. The Weibull material scale parameter has units of stress·(volume)$^{1/m}$ and should be reported using units of MPa·(m)$^{3/m}$ or GPa·(m)$^{3/m}$ if the strength-controlling flaws are distributed through the volume of the material. If the strength-controlling flaws are restricted to the surface of the specimens in a sample, then the Weibull material scale parameter should be reported using units of MPa·(m)$^{2/m}$ or GPa·(m)$^{2/m}$. For a given specimen geometry, Eq 8 and Eq 10 can be equated, which yields an expression relating σ_0 and σ_θ. Further discussion related to this issue can be found in 7.6.

3.3 For definitions of other statistical terms, terms related to mechanical testing, and terms related to advanced ceramics used in this practice, refer to Terminologies D 4392, E 456, C 1145, and E 6 or to appropriate textbooks on statistics **(2345)** .

3.4 *Symbols:*

A = specimen area (or area of effective gage section, if used).
b = gage section dimension, base of bend test specimen.
d = gage section dimension, depth of bend test specimen.
$F(x)$ = cumulative distribution function.
$f(x)$ = probability density function.
L_i = length of the inner load span for a bend test specimen.
L_o = length of the outer load span for a bend test specimen.
$\mathcal{L}$ = likelihood function.
m = Weibull modulus.
$\hat{m}$ = estimate of the Weibull modulus.
$\hat{m}_U$ = unbiased estimate of the Weibull modulus.
N = number of specimens in a sample.
P_f = probability of failure.
r = number of specimens that failed from the flaw population for which the Weibull estimators are being calculated.
t = intermediate quantity defined by Eq 27, used in calculation of confidence bounds.
V = specimen volume (or volume of effective gage section, if used).
X = random variable.
x = realization of a random variable X.
β = Weibull scale parameter.
ϵ = stopping tolerance in the computer algorithm MAXL.
$\hat{\mu}$ = estimate of mean strength.
σ = uniaxial tensile stress.
σ_i = maximum stress in the ith test specimen at failure.
σ_j = maximum stress in the jth test specimen at failure.
σ_O = Weibull material scale parameter (strength relative to unit size) defined in Eq 10.
σ_θ = Weibull characteristic strength (associated with a test specimen) defined in Eq 8.
$\hat{\sigma}_O$ = estimate of the Weibull material scale parameter.
$\hat{\sigma}_\theta$ = estimate of the Weibull characteristic strength.

4. Summary of Practice

4.1 This practice enables the experimentalist to estimate Weibull distribution parameters from failure data. Begin by performing a fractographic examination of each failed specimen (optional, but highly recommended) in order to characterize fracture origins. Usually discrete fracture origins can be grouped by flaw distributions. Screen the data associated with each flaw distribution for outliers. Compute estimates of the biased Weibull modulus and Weibull characteristic strength. If necessary, compute the estimate of the mean strength. If all failures originate from a single flaw distribution, compute an unbiased estimate of the Weibull modulus and compute confidence bounds for both the estimated Weibull modulus and the estimated Weibull characteristic strength. Prepare a graphical representation of the failure data along with a test report.

5. Significance and Use

5.1 Advanced ceramics usually display a linear stress-strain behavior to failure. Lack of ductility combined with flaws that have various sizes and orientations leads to scatter in failure strength. Strength is not a deterministic property but instead reflects an intrinsic fracture toughness and a distribution (size and orientation) of flaws present in the material. This practice is applicable to brittle monolithic ceramics that fail as a result of catastrophic propagation of flaws present in the material. This practice is also applicable to composite ceramics that do not exhibit any appreciable bilinear or nonlinear deformation behavior. In addition, the composite must contain a sufficient quantity of uniformly distributed reinforcements such that the material is effectively homogeneous. Whisker-toughened ceramic composites may be representative of this type of material.

5.2 Two- and three-parameter formulations exist for the Weibull distribution. This practice is restricted to the two-parameter formulation. An objective of this practice is to obtain point estimates of the unknown parameters by using well-defined functions that incorporate the failure data. These functions are referred to as estimators. It is desirable that an estimator be consistent and efficient. In addition, the estimator should produce unique, unbiased estimates of the distribution parameters **(6)**. Different types of estimators exist, including moment estimators, least-squares estimators, and maximum likelihood estimators. This practice details the use of maximum likelihood estimators due to the efficiency and the ease of application when censored failure populations are encountered.

5.3 Tensile and flexural specimens are the most commonly used test configurations for advanced ceramics. The observed strength values are dependent on specimen size and geometry. Parameter estimates can be computed for a given specimen geometry ($\hat{m}$, $\hat{\sigma}_{\theta}$), but it is suggested that the parameter estimates be transformed and reported as material-specific parameters ($\hat{m}$, $\hat{\sigma}_0$). In addition, different flaw distributions (for example, failures due to inclusions or machining damage) may be observed, and each will have its own strength distribution parameters. The procedure for transforming parameter estimates for typical specimen geometries and flaw distributions is outlined in 7.6.

5.4 Many factors affect the estimates of the distribution parameters. The total number of test specimens plays a significant role. Initially, the uncertainty associated with parameter estimates decreases significantly as the number of test specimens increases. However, a point of diminishing returns is reached when the cost of performing additional strength tests may not be justified. This suggests that a practical number of strength tests should be performed to obtain a desired level of confidence associated with a parameter estimate. The number of specimens needed depends on the precision required in the resulting parameter estimate. Details relating to the computation of confidence bounds (directly related to the precision of the estimate) are presented in 8.3 and 8.4.

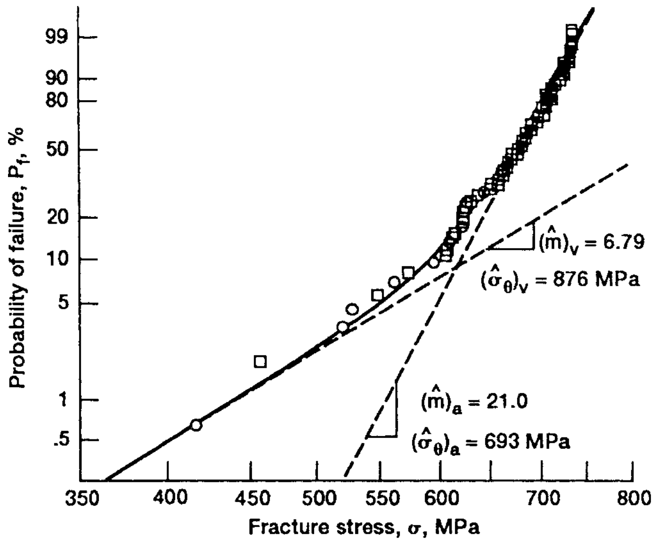
6. Outlying Observations

6.1 Before computing the parameter estimates, the data should be screened for outlying observations (outliers). An outlying observation is one that deviates significantly from other observations in the sample. It should be understood that an apparent outlying observation may be an extreme manifestation of the variability of the strength of an advanced ceramic. If this is the case, the data point should be retained and treated as any other observation in the failure sample. However, the outlying observation may be the result of a gross deviation from prescribed experimental procedure or an error in calculating or recording the numerical value of the data point in question. When the experimentalist is clearly aware that a gross deviation from the prescribed experimental procedure has occurred, the outlying observation may be discarded, unless the observation can be corrected in a rational manner. The procedures for dealing with outlying observations are detailed in Practice E 178.

7. Maximum Likelihood Parameter Estimators for Competing Flaw Distributions

7.1 This practice outlines the application of parameter estimation methods based on the maximum likelihood technique. This technique has certain advantages, especially when parameters must be determined from censored failure populations. When a sample of test specimens yields two or more distinct flaw distributions, the sample is said to contain censored data, and the associated methods for censored data must be employed. Fractography (see Section 9) should be used to determine whether multiple flaw distributions are present. The methods described in this practice include censoring techniques that apply to multiple concurrent flaw distributions. However, the techniques for parameter estimation presented in this practice are not directly applicable to data sets that contain exclusive or compound multiple flaw distributions **(7)**. The parameter estimates obtained using the maximum likelihood technique are unique (for a two-parameter Weibull distribution), and as the size of the sample increases, the estimates statistically approach the true values of the population.

7.2 This practice allows failure to be controlled by multiple flaw distributions. Advanced ceramics typically contain two or more active flaw distributions each with an independent set of parameter estimates. The censoring techniques presented herein require positive confirmation of multiple flaw distributions, which necessitates fractographic examination to characterize the fracture origin in each specimen. Multiple flaw distributions may be further evidenced by deviation from the linearity of the data from a single Weibull distribution (for example, Fig. 1). However, since there are many exceptions,



NOTE 1—The boxes refer to surface flaws; the circles refer to volume flaws.

FIG. 1 Example—Failure Data in Section 10.2

observations of approximately linear behavior should not be considered sufficient reason to conclude that only a single flaw distribution is active.

7.2.1 For data sets with multiple active flaw distributions where one flaw distribution (identified by fractographic analysis) occurs in a small number of specimens, it is sufficient to report the existence of this flaw distribution (and the number of occurrences), but it is not necessary to estimate Weibull parameters. Estimates of the Weibull parameters for this flaw distribution would be potentially biased with wide confidence bounds (neither of which could be quantified through use of this practice). However, special note should be made in the report if the occurrences of this flaw distribution take place in the upper or lower tail of the sample strength distribution.

7.3 The application of the censoring techniques presented in this practice can be complicated by the presence of test specimens that fail from extraneous flaws, fractures that originate outside the effective gage section, and unidentified fracture origins. If these complications arise, the strength data from these specimens should generally not be discarded. Strength data from specimens with fracture origins outside the effective gage section **(8)**, and specimens with fractures that originate from extraneous flaws should be censored; and the maximum likelihood methods presented in this practice are applicable.

NOTE 1—In this standard the gage section in four-point flexure is taken to mean the region between the two outer loading rollers.

7.3.1 Specimens with unidentified fracture origins sometimes occur. It is imperative that the number of unidentified fracture origins, and how they were classified, be stated in the test report. This practice recognizes four options the experimentalist can pursue when unidentified fracture origins are encountered during fractographic examinations. The situation may arise where more than one option will be used within a single data set. Specimens with unidentified fracture origins can be:

7.3.1.1 *Option a*—Assigned a previously identified flaw distribution using inferences based on all available fractographic information,

7.3.1.2 *Option b*—Assigned the same flaw distribution as that of the specimen closest in strength,

7.3.1.3 *Option c*—Assigned a new and as yet unspecified flaw distribution, and

7.3.1.4 *Option d*—Be removed from the sample.

NOTE 2—The user is cautioned that the use of any of the options outlined in 7.3.1 for the classification of specimens with unidentified fracture origins may create a consistent bias error in the parameter estimates. In addition, the magnitude of the bias cannot be determined by the methods presented in 8.2

7.3.2 A discussion of the appropriateness of each option in 7.3.1 is given in Appendix X2. If the strength data and the resulting parameter estimates are used for component design, the engineer must consult with the fractographer before and after performing the fractographic examination. Considerable judgement may be needed to identify the correct option. Whenever partial fractographic information is available, 7.3.1.1 is strongly recommended, especially if the data are used for component design. Conversely, 7.3.1.4 is not recommended by this practice unless there is overwhelming justification.

7.4 The likelihood function for the two-parameter Weibull distribution of a censored sample is defined by the following equation **(9)**:

$$\mathcal{L}\left\{\prod_{i=1}^{r}\left(\frac{\widehat{m}}{\widehat{\sigma}_\theta}\right)\left(\frac{\sigma_i}{\widehat{\sigma}_\theta}\right)^{\hat{m}-1} exp\left[-\left(\frac{\sigma_i}{\widehat{\sigma}_\theta}\right)^{\hat{m}}\right]\right\}\prod_{j=r+1}^{N} exp\left[-\left(\frac{\sigma_j}{\widehat{\sigma}_\theta}\right)^{\hat{m}}\right] \quad (12)$$

This expression is applied to a sample where two or more active concurrent flaw distributions have been identified from fractographic inspection. For the purpose of the discussion here, the different distributions will be identified as flaw Types A, B, C, etc. When Eq 12 is used to estimate the parameters associated with the A flaw distribution, then r is the number of specimens where Type A flaws were found at the fracture origin, and i is the associated index in the first summation. The second summation is carried out for all other specimens not failing from type A flaws (that is, Type B flaws, Type C flaws, etc.). Therefore, the sum is carried out from ($j = r + 1$) to N (the total number of specimens) where j is the index in the second summation. Accordingly, σ_i and σ_j are the maximum stress in the ith and jth test specimen at failure. The parameter estimates (the Weibull modulus $\hat{m}$ and the characteristic strength $\hat{\sigma}_\theta$) are determined by taking the partial derivatives of the logarithm of the likelihood function with respect to $\hat{m}$ and $\hat{\sigma}_\theta$ and equating the resulting expressions to zero. Note that $\hat{\sigma}_\theta$ is a function of specimen geometry and the estimate of the Weibull modulus. Expressions that relate $\hat{\sigma}_\theta$ to the Weibull material scale parameter $\hat{\sigma}_0$ for typical specimen geometries are given in 7.6. Finally, the likelihood function for the two-parameter Weibull distribution for a single-flaw population is defined by the following equation:

$$\mathcal{L} = \prod_{i=1}^{N}\left(\frac{\hat{m}}{\widehat{\sigma}_\theta}\right)\left(\frac{\sigma_i}{\widehat{\sigma}_\theta}\right)^{\hat{m}-1} exp\left[-\left(\frac{\sigma_i}{\widehat{\sigma}_\theta}\right)^{\hat{m}}\right] \quad (13)$$

where r was taken equal to N in Eq 12.

7.5 The system of equations obtained by maximizing the log likelihood function for a censored sample is given by the following equations **(10)**:

$$\frac{\sum_{i=1}^{N}(\sigma_i)^{\hat{m}}\, 1n(\sigma_i)}{\sum_{i=1}^{N}(\sigma_i)^{\hat{m}}} - \frac{1}{r}\sum_{i=1}^{r} 1n(\sigma_i) - \frac{1}{\hat{m}} = 0 \qquad (14)$$

and

$$\hat{\sigma}_\theta = \left[\left(\sum_{i=1}^{N} (\sigma_i)^{\hat{m}}\right)\frac{1}{r}\right]^{1/\hat{m}} \qquad (15)$$

where:

r = number of failed specimens from a particular group of a censored sample.

When a sample does not require censoring, r is replaced by N in Eq 14 and Eq 15. Eq 14 is solved first for $\hat{m}$. Subsequently $\hat{\sigma}_\theta$ is computed from Eq 15. Obtaining a closed-form solution of Eq 14 for $\hat{m}$ is not possible. This expression must be solved numerically. When there are multiple active flaw populations, Eq 14 and Eq 15 must be solved for each flaw population. A computer algorithm (entitled MAXL) that calculates the root of Eq 14 is presented as a convenience in Appendix X1.

7.6 The numerical procedure in accordance with 7.5 yields parameter estimates of the Weibull modulus ($\hat{m}$) and the characteristic strength ($\hat{\sigma}_\theta$). Since the characteristic strength also reflects specimen geometry and stress gradients, this standard suggests reporting the estimated Weibull material scale parameter $\hat{\sigma}_0$.

7.6.1 The following equation defines the relationship between the parameters for tensile specimens:

$$(\hat{\sigma}_0)_V = (V)^{1/(\hat{m})v}(\hat{\sigma}_\theta)_V \qquad (16)$$

where V is the volume of the uniform gage section of the tensile specimen, and the fracture origins are spatially distributed strictly within this volume. The gage section of a tensile specimen is defined herein as the central region of the test specimen with the smallest constant cross-sectional area. However, the experimentalist may include transition regions and the shank regions of the specimen if the volume (or area) integration defined by Eq 10 is analyzed properly. This procedure is discussed in 7.6.3. If the transition region or the shank region, or both, are included in the integration, Eq 16 is not applicable. For tensile specimens in which fracture origins are spatially distributed strictly at the surface of the specimens tested, the following equation applies:

$$(\hat{\sigma}_0)_A = (A)^{1/(\hat{m})A}(\hat{\sigma}_\theta)_A \qquad (17)$$

where A = surface area of the uniform gage section.

7.6.2 For flexural specimen geometries, the relationships become more complex **(11)**. The following relationship is based on the geometry of a flexural specimen found in Fig. 2. For fracture origins spatially distributed strictly within both the volume of a flexural specimen and the outer load span, the following equation applies:

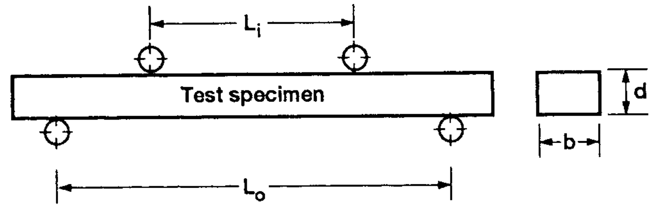


FIG. 2 Flexural Specimen Geometry

$$(\hat{\sigma}_0)_V = (\hat{\sigma}_\theta)_V \left| \frac{V\left[\left(\frac{L_i}{L_o}\right)(\hat{m})_v + 1\right]}{2[(\hat{m})_v + 1]^2} \right|^{1/(\hat{m})V} \qquad (18)$$

where:

L_i = length of the inner load span,
L_o = length of the outer load span,
V = volume of the gage section defined by the following expression:

$$V = b\, d\, L_o \qquad (19)$$

and:

b,d = dimensions identified in Fig. 2.

For fracture origins spatially distributed strictly at the surface of a flexural specimen and within the outer load span, the following equation applies:

$$(\hat{\sigma}_0)_A = (\hat{\sigma}_\theta)_A \left[L_o\left(\frac{d}{(\hat{m})_A + 1} + b\right)\left(\frac{\left(\frac{L_i}{L_o}\right)(\hat{m})_A + 1}{(\hat{m})_A + 1}\right)\right]^{1/(\hat{m})A} \qquad (20)$$

7.6.3 Test specimens other than tensile and flexure specimens may be utilized. Relationships between the estimate of the Weibull characteristic strength and the Weibull material scale parameter for any specimen configuration can be derived by equating the expressions defined by Eq 8 and Eq 10 with the modifications that follow. Begin by replacing σ (an applied uniaxial tensile stress) in Eq 8 with σ_{max}, which is defined as the maximum tensile stress within the test specimen of interest. Thus:

$$P_f = 1 - exp\left[-\left(\frac{\sigma_{max}}{\sigma_\theta}\right)^m \right] \qquad (21)$$

Also perform the integration given in Eq 10 such that

$$P_f = 1 - exp\left[-kV\left(\frac{\sigma_{max}}{\sigma_0}\right)^m \right] \qquad (22)$$

where k is a dimensionless constant that accounts for specimen geometry and stress gradients. Note that in general, k is a function of the estimated Weibull modulus m, and is always less than or equal to unity. The product (kV) is often referred to as the effective volume (with the designation V_E). The effective volume can be interpreted as the size of an equivalent uniaxial tensile specimen that has the same risk of rupture as the test specimen or component. As the term implies, the product represents the volume of material subject to a uniform uniaxial tensile stress **(12)**. Setting Eq 21 and Eq 22 equal to one another yields the following expression:

$$(\hat{\sigma}_0)_V = (kV)^{1/(\hat{m})V}(\hat{\sigma}_\theta)_V \qquad (23)$$

Thus, for an arbitrary test specimen, the experimentalist evaluates the integral identified in Eq 10 for the effective volume (*kV*), and utilizes Eq 23 to obtain the estimated Weibull material scale parameter $\hat{\sigma}_0$. A similar procedure can be adopted when fracture origins are spatially distributed at the surface of the test specimen.

7.7 An objective of this practice is the consistent representation of strength data. To this end, the following procedure is the recommended graphical representation of strength data. Begin by ranking the strength data obtained from laboratory testing in ascending order, and assign to each a ranked probability of failure P_f according to the estimator as follows:

$$P_f(\sigma_i) = \frac{i - 0.5}{N} \tag{24}$$

where
N = number of specimens, and
i = ith datum.

Compute the natural logarithm of the ith failure stress, and the natural logarithm of the natural logarithm of $[1/(1 - P_f)]$ (that is, the double logarithm of the quantity in brackets), where P_f is associated with the ith failure stress.

7.8 Create a graph representing the data as shown in Fig. 1. Plot $\ln\{\ln[1/(1 - P_f)]\}$ as the ordinate, and $\ln(\sigma)$ as the abscissa. A typical ordinate scale assumes values from +2 to −6. This approximately corresponds to a range in probability of failure from 0.25 to 99.9 %. The ordinate axis must be labeled as probability of failure P_f, as depicted in Fig. 1. Similarly, the abscissa must be labeled as failure stress (flexural, tensile, etc.), preferably using units of megapascals or gigapascals.

7.9 Included on the plot should be a line (two or more lines for concurrent flaw distributions) whose position is fixed by the estimates of the Weibull parameters. The line is defined by the following mathematical equation:

$$P_f = 1 - exp\left[-\left(\frac{\sigma}{\hat{\sigma}_\theta}\right)\hat{m}\right] \tag{25}$$

The slope of the line, which is the estimate of the Weibull modulus $\hat{m}$, should be identified, as shown in Fig. 1. The estimate of the characteristic strength σ_θ should also be identified. This corresponds to a P_f of 63.2 %, or a value of zero for $\ln \| ln[1/1(1 - P_f)] \|$. A test report (that is, a data sheet) that details the type of material characterized, the test procedure (preferably designating an appropriate standard), the number of failed specimens, the flaw type, the maximum likelihood estimates of the Weibull parameters, the unbiasing factor, and the information that allows the construction of 90 % confidence bounds is depicted in Fig. 3. This data sheet should accompany the graph to provide a complete representation of the failure data. Insert a column on the graph (in any convenient location), or alternatively provide a separate table that identifies the individual strength values in ascending order as shown in Fig. 4.**(13)** This will permit other users to perform alternative analyses (for example, future implementations of bias correction or confidence bounds, or both, on multiple flaw populations). In addition, the experimentalist should include a separate sketch of the specimen geometry that includes all pertinent dimensions. An estimate of mean strength can also be depicted in the graph. The estimate of mean strength |gm is calculated by using the arithmetic mean as the estimator in the following equation:

$$\hat{\mu} = \left(\sum_{i=1}^{N} \sigma_i\right)\left(\frac{1}{N}\right) \tag{26}$$

Note that this estimate of the mean strength is not appropriate for samples with multiple failure populations.

8. Unbiasing Factors and Confidence Bounds

8.1 Paragraphs 8.2 through 8.4 outline methods to correct for statistical bias errors in the estimated Weibull parameters and outlines methods to calculate confidence bounds. The procedures described herein to correct for statistical bias errors and to compute confidence bounds are appropriate only for data sets where all failures originate from a single flaw population (that is, an uncensored sample). Procedures for bias correction and confidence bounds in the presence of multiple active flaw populations are not well developed at this time. Note that the statistical bias associated with the estimator $\hat{\sigma}_\theta$ is minimal (<0.3 % for 20 test specimens, as opposed to ≃7 % bias for m with the same number of specimens). Therefore, this practice allows the assumption that $\hat{\sigma}_\theta$ is an unbiased estimator of the true population parameter. The parameter estimate of the Weibull modulus ($\hat{m}$) generally exhibits statistical bias. The amount of statistical bias depends on the number of specimens in the sample. An unbiased estimate of m shall be obtained by multiplying $\hat{m}$ by unbiasing factors **(14)**. This procedure is discussed in the following sections. Statistical bias associated with the maximum likelihood estimators presented in this practice can be reduced by increasing the sample size.

8.2 An unbiased estimator produces nearly zero statistical bias between the value of the true parameter and the point estimate. The amount of deviation can be quantified either as a percent difference or with unbiasing factors. In keeping with the accepted practice in the open literature, this practice quantifies statistical bias through the use of unbiasing factors, denoted here as *UF*. Depending on the number of specimens in a given sample, the point estimate of the Weibull modulus $\hat{m}$ may exhibit significant statistical bias. An unbiased estimate of the Weibull modulus (denoted as $\hat{m}_U$) is obtained by multiplying the biased estimate with appropriate unbiasing factor. Inbiasing factors for $\hat{m}$ are listed in Table 1. The example in 11.3 demonstrates the use of Table 1 in correcting a biased estimate of the Weibull modulus. As a final note, this procedure is not appropriate for censored samples. The theoretical approach was developed for uncensored samples where $r = N$.

8.3 Confidence bounds quantify the uncertainty associated with a point estimate of a population parameter. The size of the confidence bounds for maximum likelihood estimates of both Weibull parameters will diminish with increasing sample size. The values used to construct confidence bounds are based on percentile distributions obtained by Monte Carlo simulation. For example, the 90 % confidence bound on the Weibull modulus is obtained from the 5 and 95 percentile distributions of the ratio of $\hat{m}$ to the true population value m. For the point

TEST REPORT

Weibull Parameters Calculated Using Maximum Likelihood Estimators

Material: ______
Test method: ______
Specimen size: ______
Specimens from:
Single billet ☐ ______
Multiple billets ☐ ______
Component(s) ☐ ______
Separately made ☐ ______
Total number of specimens: ______

FLAW POPULATION 1

Number of specimens: ______
Flaw identity: ______
Spatial dist. ☐ Volume
☐ Surface
☐ ______
Estimates:
$\hat{m}$ = ______
$\hat{\sigma}_\theta$ = ______
$\hat{\sigma}_0$ = ______ (Weibull scale parameter)

FLAW POPULATION 2

Number of specimens: ______
Flaw identity: ______
Spatial dist. ☐ Volume
☐ Surface
☐ ______
Estimates:
$\hat{m}$ = ______
$\hat{\sigma}_\theta$ = ______
$\hat{\sigma}_0$ = ______ (Weibull scale parameter)

FLAW POPULATION 3

Number of specimens: ______
Flaw identity: ______
Spatial dist. ☐ Volume
☐ Surface
☐ ______
Estimates:
$\hat{m}$ = ______
$\hat{\sigma}_\theta$ = ______
$\hat{\sigma}_0$ = ______ (Weibull scale parameter)

Complete the following and report the numbers below if only one flaw population exists:

Correct m for bias (Table 1)
UF = ______ $\hat{m}_u = \hat{m}$ UF = ______

90% Confidence bounds:
(Note: Use $\hat{m}$ below, **not** $\hat{m}_u$.)

REPORT THESE

m (Table 2)

$q_{0.05}$ = ______ $\hat{m}_{upper} = \hat{m}/q_{0.05}$ = ______
$q_{0.95}$ = ______ $\hat{m}_{lower} = \hat{m}/q_{0.95}$ = ______

σ_θ (Table 3)

$t_{0.05}$ = ______ $\hat{\sigma}_{\theta,upper} = \hat{\sigma}_\theta \exp(-t_{0.05}/\hat{m})$ = ______
$t_{0.95}$ = ______ $\hat{\sigma}_{\theta,lower} = \hat{\sigma}_\theta \exp(-t_{0.95}/\hat{m})$ = ______

How were unidentified specimens treated?
Number of unidentified specimens: ______
☐ Identity estimated by extrapolating fractography
☐ Identity assigned arbitrarily to be same as the nearest strength datum
☐ Assumed to belong to a distinct population
☐ Discarded as random events

FIG. 3 Sample Test Report

estimate of the Weibull modulus, the normalized values ($\hat{m}/m$) necessary to construct the 90 % confidence bounds are listed in

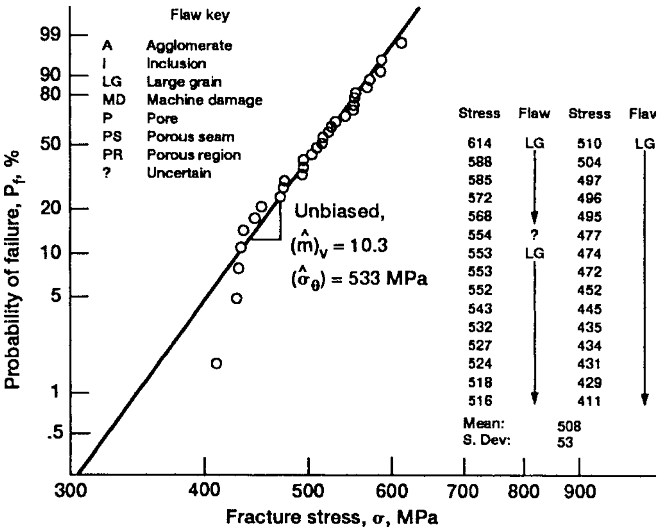


FIG. 4 Example—Failure Data with Fractography Information (13)

TABLE 1 Unbiasing Factors for the Maximum Likelihood Estimate of the Weibull Modulus

Number of Specimens, *N*	Unbiasing Factor, *UF*	Number of Specimens, *N*	Unbiasing Factor, *UF*
5	0.700	42	0.968
6	0.752	44	0.970
7	0.792	46	0.971
8	0.820	48	0.972
9	0.842	50	0.973
10	0.859	52	0.974
11	0.872	54	0.975
12	0.883	56	0.976
13	0.893	58	0.977
14	0.901	60	0.978
15	0.908	62	0.979
16	0.914	64	0.980
18	0.923	66	0.980
20	0.931	68	0.981
22	0.938	70	0.981
24	0.943	72	0.982
26	0.947	74	0.982
28	0.951	76	0.983
30	0.955	78	0.983
32	0.958	80	0.984
34	0.960	85	0.985
36	0.962	90	0.986
38	0.964	100	0.987
40	0.966	120	0.990

TABLE 2 Normalized Upper and Lower Bounds on the Maximum Likelihood Estimate of the Weibull Modulus—90 % Confidence Interval

Number of Specimens, *N*	$q_{0.05}$	$q_{0.95}$	Number of Specimens, *N*	$q_{0.05}$	$q_{0.95}$
5	0.683	2.779	42	0.842	1.265
6	0.697	2.436	44	0.845	1.256
7	0.709	2.183	46	0.847	1.249
8	0.720	2.015	48	0.850	1.242
9	0.729	1.896	50	0.852	1.235
10	0.738	1.807	52	0.854	1.229
11	0.745	1.738	54	0.857	1.224
12	0.752	1.682	56	0.859	1.218
13	0.759	1.636	58	0.861	1.213
14	0.764	1.597	60	0.863	1.208
15	0.770	1.564	62	0.864	1.204
16	0.775	1.535	64	0.866	1.200
17	0.779	1.510	66	0.868	1.196
18	0.784	1.487	68	0.869	1.192
19	0.788	1.467	70	0.871	1.188
20	0.791	1.449	72	0.872	1.185
22	0.798	1.418	74	0.874	1.182
24	0.805	1.392	76	0.875	1.179
26	0.810	1.370	78	0.876	1.176
28	0.815	1.351	80	0.878	1.173
30	0.820	1.334	85	0.881	1.166
32	0.824	1.319	90	0.883	1.160
34	0.828	1.306	95	0.886	1.155
36	0.832	1.294	100	0.888	1.150
38	0.835	1.283	110	0.893	1.141
40	0.839	1.273	120	0.897	1.133

Table 2. The example in 10.3 demonstrates the use of Table 2 in constructing the upper and lower bounds in $\hat{m}$. Note that the statistical biased estimate of the Weibull modulus must be used here. Again, this procedure is not appropriate for censored statistics.

8.4 Confidence bounds can be constructed for the estimated Weibull characteristic strength. However, the percentile distributions needed to construct the bounds do not involve the same normalized ratios or the same tables as those used for the Weibull modulus. Define the function as follows:

$$t = \hat{m}\, ln(\hat{\sigma}_\theta / \sigma_\theta) \tag{27}$$

The 90 % confidence bound on the characteristic strength is obtained from the 5 and 95 percentile distributions of t. For the point estimate of the characteristic strength, these percentile distributions are listed in Table 3. The example in 10.3 demonstrates the use of Table 3 in constructing upper and lower bounds on $\hat{\sigma}_\theta$. Note that the biased estimate of the Weibull modulus must be used here. Again, this procedure is not appropriate for censored statistics. Note that Eq 27 is not

TABLE 3 Normalized Upper and Lower Bounds on the Function *t*—90 % Confidence Interval

Number of Specimens, *N*	$t_{0.05}$	$t_{0.95}$	Number of Specimens, *N*	$t_{0.05}$	$t_{0.95}$
5	−1.247	1.107	42	−0.280	0.278
6	−1.007	0.939	44	−0.273	0.271
7	−0.874	0.829	46	−0.266	0.264
8	−0.784	0.751	48	−0.260	0.258
9	−0.717	0.691	50	−0.254	0.253
10	−0.665	0.644	52	−0.249	0.247
11	−0.622	0.605	54	−0.244	0.243
12	−0.587	0.572	56	−0.239	0.238
13	−0.557	0.544	58	−0.234	0.233
14	−0.532	0.520	60	−0.230	0.229
15	−0.509	0.499	62	−0.226	0.225
16	−0.489	0.480	64	−0.222	0.221
17	−0.471	0.463	66	−0.218	0.218
18	−0.455	0.447	68	−0.215	0.214
19	−0.441	0.433	70	−0.211	0.211
20	−0.428	0.421	72	−0.208	0.208
22	−0.404	0.398	74	−0.205	0.205
24	−0.384	0.379	76	−0.202	0.202
26	−0.367	0.362	78	−0.199	0.199
28	−0.352	0.347	80	−0.197	0.197
30	−0.338	0.334	85	−0.190	0.190
32	−0.326	0.323	90	−0.184	0.185
34	−0.315	0.312	95	−0.179	0.179
36	−0.305	0.302	100	−0.174	0.175
38	−0.296	0.293	110	−0.165	0.166
40	−0.288	0.285	120	−0.158	0.159

applicable for developing confidence bounds on $\hat{\sigma}_0$, therefore the confidence bounds on $\hat{\sigma}_\theta$ should not be converted through the use of Eq 8 and Eq 10.

9. Fractography

9.1 Fractographic examination of each failed specimen is highly recommended in order to characterize the fracture origins. The strength of advanced ceramics is often limited by discrete fracture origins that may be intrinsic or extrinsic to the material. Porosity, agglomerates, inclusions, and atypical large grains would be considered intrinsic fracture origins. Extrinsic fracture origins are typically on the surface of the specimen and are the result of contact stresses, impact events, or adverse environment. When the means are available to the experimentalist, fractographic methods should be used to locate, identify, and classify the strength limiting fracture origin causing catastrophic failure in an advanced ceramic test specimen. Moreover, for the purpose of parameter estimation, each classification of fracture origin must be identified as a surface fracture origin or a volume fracture origin in order to use the expressions given in 7.6. The classification shall be based on the spatial distribution of a given flaw type (that is, volume-distributed pores versus surface-distributed machining damage) and not the specific location of a given flaw in a particular specimen. Thus, there may exist several classifications of fracture origins within the volume (or surface area) of the test specimens in a sample. It should be clearly indicated on the test report (Fig. 3) if a fractographic analysis is not performed. The experimentalist is urged to follow the guidelines established in Practice C 1322 or the MIL—HDBK–790 concerning fractography cited in 2.2.

9.2 *Optional*—Perform a fractographic analysis and label each datum with a symbol identifying the type of fracture origin. This can either be a word, an abbreviation, or a different symbol for each type of fracture origin, as depicted in Fig. 4. For example, the abbreviations in LG in Fig. 4 represents failure due to a large grain.

10. Examples

10.1 For the first example, consider the failure data in Table 4. The data represent four-point (¼point) flexural specimens fabricated from HIP'ed (hot isostatically pressed) silicon carbide **(15)**. The solution of Eq 14 requires an iterative numerical scheme. Using the computer algorithm MAXL (see Appendix X1), a parameter estimate of $\hat{m}$ = 6.48 was obtained. (Note that an unbiased value of $\hat{m}$ = 6.38 is shown in Fig. 5; See 10.3 and Eq 31.) Subsequent solution of Eq 15 yields a value of $\hat{\sigma}_\theta$= 556 MPa. These values for the Weibull parameters were generated by assuming a unimodal failure sample with no censoring (that is, $r = N$). Fig. 5 depicts the individual failure data and a curve based on the estimated values of the parameters. Next, assuming that the failure origins were surface distributed and then inserting the estimated value of $\hat{m}$ and $\hat{\sigma}_\theta$ into Eq 20 along with the specimen geometry (that is, L_o = 40 mm, L_i= 20 mm, d = 3.5 mm, and b = 4.5 mm) yields $(\hat{\sigma}_0)_a$ = 137 MPa·$(m)^{0.309}$. Note that $(\hat{\sigma}_0)_a$ has units of stress·(area)$^{1/\hat{m}}$; thus, 0.309 = (2./6.48). Alternative, if one were to assume that the failure origins were volume distributed, then the solution of Eq 18 yields $(\hat{\sigma}_0)_V$= 37.0 MPa·$(m)^{0.463}$. Note that $(\hat{\sigma}_0)_V$ has units of stress·(volume)$^{1/\hat{m}}$; thus, 0.463 = (3./6.48). The different values obtained from assuming surface and volume fracture origins underscore the necessity of conducting a fractographic analysis.

TABLE 4 Unimodal Failure Stress Data for Hipped (Hot Isostatically Pressed) Silicon Carbide—Example 1

Specimen number, N	Strength, σ_i, MPa	Specimen number, N	Strength, σ_i, MPa
1	281	41	516
2	291	42	520
3	358	43	528
4	385	44	531
5	389	45	531
6	391	46	546
7	392	47	549
8	403	48	553
9	412	49	560
10	413	50	562
11	414	51	563
12	418	52	566
13	418	53	566
14	427	54	570
15	438	55	573
16	440	56	575
17	441	57	576
18	442	58	580
19	444	59	583
20	445	60	588
21	446	61	589
22	452	62	591
23	452	63	591
24	453	64	593
25	470	65	599
26	474	66	600
27	476	67	610
28	476	68	613
29	479	69	620
30	484	70	620
31	485	71	622
32	486	72	622
33	489	73	640
34	492	74	649
35	493	75	657
36	496	76	660
37	506	77	664
38	512	78	674
39	512	79	674
40	514	80	725

10.2 Next, consider a sample that exhibits multiple active flaw distributions (see Table 5). Here each flexural test specimen was subjected to a fractographic analysis. The failure origin was identified as either a volume or a surface fracture origin, and parameter estimates were obtained by using Eq 14 and Eq 15. For the analysis with volume fracture origins, r = 13, and the calculations yielded values of $(\hat{m})_V$= 6.79 and $(\hat{\sigma}_\theta)_V$= 876 MPa. For the analysis with surface fracture origins, r = 66, and the calculations yielded values of $(\hat{m})_a$= 21.0 and $(\hat{\sigma}_\theta)_a$ = 693 MPa. For the most part, the data as plotted in Fig. 1 fall near the solid curve, which represents the combined probability of failure as follows **(16)**:

$$P_f = 1 - [1 - (P_f)_A][1 - (P_f)_V] \tag{28}$$

where $(P_f)_V$ is calculated by using the following equation:

$$(P_f)_V = 1 - exp\left[- \left(\frac{\sigma}{(\hat{\sigma}_\theta)_V}\right)^{(\hat{m})_V}\right] \tag{29}$$

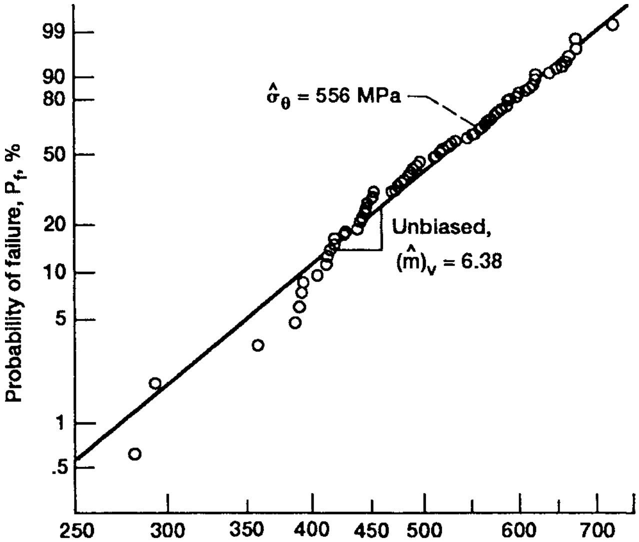


FIG. 5 Example—Failure Data in 10.1

and $(P_f)_A$ is calculated by using the following equation:

$$(P_f)_A = 1 - exp\left[- \left(\frac{\sigma}{(\hat{\sigma}_\theta)_a} \right)^{(\hat{m})A} \right] \tag{30}$$

The curve obtained from Eq 28 asymptotically approaches the intersecting straight lines that are defined by the estimated parameters and calculated from Eq 29 and Eq 30. Inserting the estimated Weibull parameters (obtained from the analysis for volume fracture origins) into Eq 18 along with the specimen geometry (L_o= 40 mm, L_i= 20 mm, d = 3.5 mm, and b = 4.5 mm) yields $(\hat{\sigma}_0)_V$= 65.6 MPa·$(m)^{0.442}$. Inserting the estimated Weibull parameters (obtained from the analysis for surface fracture origins) into Eq 20 yields $(\hat{\sigma}_0)_a$= 446 MPa·$(m)^{0.95}$.

10.2.1 It must be noted in this example that fractography apparently indicated that all volume failures were initiated from a single distribution of volume flaws, and that all surface failures were initiated from a single distribution of surface flaws. Often, fractography will indicate more complex situations such as two independent distributions of volume flaws (for example, inclusions of foreign material and large voids) in addition to a distribution of surface flaws. Analysis of this type of sample would be very similar to the analysis discussed in 10.1, except that Eq 14 and Eq 15 would be used three times instead of twice, and the resulting figure would include three straight lines labelled accordingly.

10.3 As an example of computing unbiased estimates of the Weibull modulus, and bounds on both the Weibull modulus and the Weibull characteristic strength, consider the unimodal failure sample presented in 10.1. The sample contained 80 specimens and the biased estimate of the Weibull modulus was determined to be $\hat{m}$ = 6.48. The unbiasing factor corresponding to this sample size is UF = 0.984, which is obtained from Table 1. Thus, the unbiased estimate of the Weibull modulus is given as follows:

$$\hat{m}^U = \hat{m} \times UF$$

$$= (6.48)(0.984)$$

$$= 6.38 \tag{31}$$

TABLE 5 Bimodal Failure Stress Data—Example 2

Number of Specimens, *N*	Strength, MPa	Fracture Origin type[A]	Number of Specimens, *N*	Strength, MPa	Fracture Origin[A]
1	416	V	41	671	S
2	458	S	42	672	S
3	520	V	43	672	S
4	527	V	44	674	S
5	546	S	45	677	S
6	561	V	46	677	S
7	572	S	47	678	S
8	595	V	48	680	S
9	604	S	49	683	S
10	604	S	50	684	S
11	609	V	51	686	S
12	612	S	52	687	S
13	614	S	53	687	S
14	621	V	54	691	S
15	622	S	55	694	S
16	622	S	56	695	S
17	622	V	57	700	S
18	622	S	58	703	S
19	625	S	59	703	S
20	626	V	60	703	S
21	631	S	61	703	S
22	640	S	62	704	S
23	643	V	63	704	S
24	649	S	64	706	S
25	650	S	65	710	S
26	652	V	66	713	S
27	655	S	67	716	S
28	657	S	68	716	S
29	657	V	69	716	S
30	660	S	70	716	S
31	660	S	71	716	S
32	662	V	72	717	S
33	662	S	73	725	S
34	662	S	74	725	S
35	664	S	75	725	S
36	664	S	76	726	S
37	664	S	77	727	S
38	666	S	78	729	S
39	669	S	79	732	S
40	671	S	. . .	. . .	. . .

[A] Volume fracture origin, *V*; surface flaw origin, *S*

The upper bound on $\hat{m}$ for this example is as follows:

$$\hat{m}_{upper} = \hat{m}/q_{0.05}$$

$$= 6.48/0.878$$

$$= 7.38 \tag{32}$$

where $q_{0.05}$ is obtained from Table 2 for a sample size of 80 failed specimens. The lower bound is as follows:

$$\hat{m}_{lower} = \hat{m}/q_{0.95}$$

$$= 6.48/1.173$$

$$= 5.52 \tag{33}$$

where $q_{0.95}$ is obtained from Table 2. Similarly, the upper bound on $\hat{\sigma}_\theta$ is as follows:

$$(\hat{\sigma}_\theta)_{upper} = \hat{\sigma}_\theta\, exp(-t_{0.05}/\hat{m})$$
$$= (556)exp(0.197/6.48)$$
$$= 573\ MPa \tag{34}$$

where $t_{0.05}$ is obtained from Table 3 for a sample size of 80 failed specimens. The lower bound on $\hat{\sigma}_\theta$ is as follows:

$$(\hat{\sigma}_\theta)_{lower} = \hat{\sigma}_\theta\, exp(-t_{0.95}/\hat{m}) = (556)exp(0.197/6.48) = 539\ MPa \quad (35)$$

where $t_{0.95}$ is also obtained from Table 3.

11. Keywords

11.1 advanced ceramics; censored data; confidence bounds; fractography; fracture origin; maximum likelihood; strength; unbiasing factors; Weibull characteristic strength; Weibull modulus; Weibull scale parameter; Weibull statistics

APPENDIXES

(Nonmandatory Information)

X1. COMPUTER ALGORITHM MAXL

X1.1 Using maximum likelihood estimators to compute estimates of the Weibull parameters requires solving Eq 14 and Eq 15 for $\hat{m}$ and $\hat{\sigma}_\theta$, respectively. The solution of Eq 15 is straightforward once the estimate of the Weibull modulus $\hat{m}$ is obtained from Eq 14. Obtaining the root of Eq 14 requires an iterative numerical solution. In this appendix, the theoretical approach is presented for the numerical solution of these equations, along with the details of a computer algorithm (optional) that can be used to solve Eq 14 and Eq 15. A flow chart of the algorithm, which is entitled MAXL, is presented in Fig. X1.1.

X1.2 The MAXL algorithm employs a Newton-Raphson technique **(17)** to find the root of Eq 14. The root of Eq 14 represents a biased estimate of the Weibull modulus. Solution of Eq 15, which depends on the *biased* value of $\hat{m}$, is effectively an *unbiased* estimate of the characteristic strength. The reader is cautioned not to correct $\hat{m}$ for bias prior to computing the characteristic strength. This would yield an incorrect value of $\hat{\sigma}_\theta$. This approach expands Eq 14 in a Taylor series about $\hat{m}_0$;

$$f(\hat{m}) = f(\hat{m}_0) + (\hat{m} - \hat{m}_0)[f'(\hat{m}_0)] + \left[\frac{(\hat{m} - \hat{m}_0)^2}{2}\right] f''(\hat{m}_0) + \ldots \quad (X1.1)$$

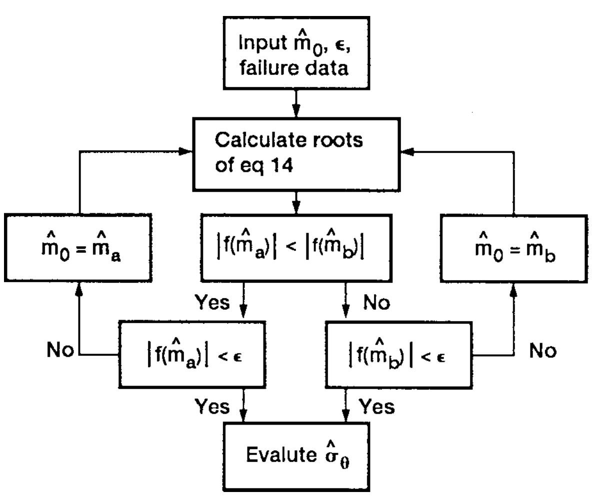


FIG. X1.1 MAXL Flow Chart

where $f(\hat{m})$ represents the right-hand side of Eq 14 and $\hat{m}_0$ is not a root of $f(\hat{m})$ but is reasonably close. Taking:

$$\Delta\hat{m} = \hat{m} - \hat{m}_0 \quad (X1.2)$$

and setting Eq X 1.1 equal to zero, then:

$$0 = f(\hat{m}_0) + (\Delta\hat{m})[f'(\hat{m}_0)] + \left[\frac{(\Delta\hat{m})^2}{2}\right] f''(\hat{m}_0) + \ldots \quad (X1.3)$$

If the Taylor series expansion is truncated after the first three terms, the resulting expression is quadratic in $\Delta\hat{m}$. The roots of the quadratic form of Eq X 1.3 are as follows:

$$\Delta\hat{m}_{a,b} = -\left[\frac{f'(\hat{m})}{f''(\hat{m})}\right] \pm \left[\left(\frac{f'(\hat{m})}{f''(\hat{m})}\right)^2 - 2\left(\frac{f(\hat{m})}{f''(\hat{m})}\right)\right]^{1/2} \quad (X1.4)$$

After obtaining $\Delta\hat{m}_{a,b}$ and knowing $\hat{m}_0$, Eq X 1.2 is then solved for two values of m that represent improved (better than $\hat{m}_0$) estimates of the roots of $f(\hat{m})$, thus

$$\hat{m}_a = \hat{m}_0 + \Delta\hat{m}_a \quad (X1.5)$$

and

$$\hat{m}_b = \hat{m}_0 + \Delta\hat{m}_b \quad (X1.6)$$

Eq 14 is evaluated with both values of $\hat{m}$, and the quantity that yields a smaller functional value is accepted as the updated estimate. This updated value of $\hat{m}$ replaces $\hat{m}_0$ in Eq X 1.4, and the next iteration is performed. The iterative procedure is terminated when the functional evaluation of Eq 14 becomes less than some predetermined tolerance ϵ.

X1.3 The following variable name list is provided as a convenience for interpreting the source code of the algorithm MAXL: DF, DDF—first and second derivatives with respect to $\hat{m}$ of Eq 14.
EPS—predetermined convergence criterion.
F—function defined in Eq 14.
NLIM—maximum numbers of iterations allowed in determining the root of Eq X 1.3.
NSUSP—number of suspended (or censored) data (<*NT*).
NT—number of failure stresses.
ST—failure stress; an argument passed to MAXL as input.
STNORM—the largest failure stress; used to normalize all

failure stresses to prevent computational overflows.
MO—updated value of $\hat{m}$.
MA, MB—values of the roots $\hat{m}_a$ and $\hat{m}_b$.
WCS—estimated Weibull characteristic strength.
WMT—maximum likelihood estimate of the Weibull modulus.

X1.4 A listing of the FORTRAN source code of the algorithm MAXL is given in Fig. X1.2, Fig. X1.3, and Fig. X1.4.

```
C
C   ******************************************************************
C   *                                                                *
C   *                         PROGRAM MAXL                           *
C   *                                                                *
C   *       THIS PROGRAM CALCULATES TWO PARAMETER MAXIMUM            *
C   *       LIKELIHOOD ESTIMATES FROM FAILURE DATA WITH AN           *
C   *       ASSUMED UNDERLYING WEIBULL DISTRIBUTION.  THE            *
C   *       ALGORITHM USES A NONLINEAR NEWTON-RAPHSON METHOD,        *
C   *       AND ACCOMODATES CENSORED DATA.                           *
C   *                                                                *
C   *       REFERENCES:  "ADVANCED CALCULUS FOR APPLICATIONS"        *
C   *                     by HILDEBRAND                              *
C   *                     PRENTICE-HALL, INC.; 1962                  *
C   *                                                                *
C   *                    "APPLIED LIFE DATA ANALYSIS"                *
C   *                     by NELSON                                  *
C   *                     WILEY & SONS INC.; 1982                    *
C   *                                                                *
C   ******************************************************************
C
      IMPLICIT REAL *8(A-H,O-Z)
      DOUBLE PRECISION ST(1000),ST1(1000)
      DOUBLE PRECISION M0, MA, MB, M1
      COMMON /DATA/ NFAIL, SUM1, NT, ST, ZERO, ONE
      ZERO = 0.D0
      ONE = 1.D0
      TWO = 2.D0
      EPS = 5.0D-10
      NLIM = 500
      M0 = 10.0
C
C   --- READ THE FAILURE DATA USING FREE FORMATS;
C        FILE CONTAINING FAILURE DATA IS ALLOCATED TO UNIT 8
C
      DO 10 I = 1,1000
        ST(I) = ZERO
        ST1(I) = ZERO
   10 CONTINUE
      STNORM = ZERO
      READ(8,*) NT
      READ(8,*) NSUSP
      PRINT*,NT,NSUSP
      NFAIL = NT - NSUSP
      DO 20 I = 1,NT
        READ(8,*) ST(I)
C        PRINT*,I,ST(I)
        STNORM = DMAX1(STNORM,ST(I))
   20 CONTINUE
C
C   --- NORMALIZE FAILURE DATA WITH LARGEST VALUE
C
      DO 30 I = 1,NT
        ST(I) = ST(I)/STNORM
C        PRINT*,I,ST(I)
   30 CONTINUE
C
      SUM1 = ZERO
      PRINT*,NFAIL
      DO 40 I = 1,NFAIL
        READ(8,*) ST1(I)
        ST1(I) = ST1(I)/STNORM
        PRINT*,I,ST1(I)
        SUM1 = SUM1 + DLOG(ST1(I))
   40 CONTINUE
C
C   --- THE FUNCTION F IS DEFINED BY EQ 14 OF ASTM STANDARD C 1239
C
C   --- EVALUATE F(M0) AND THE ASSOCIATED SUMS WHICH ARE USED TO CALCULATE
C        THE DERIVATIVES OF F WITH RESPECT TO M
C
```

FIG. X1.2 FORTRAN Source Code of the Algorithm MAXL

```
      CALL SUM (M0, SUM2, SUM3, F)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C    *********************************************************************
C    *                    NEWTON-RAPHSON ROOT SOLVER                     *
C    *********************************************************************
C
C    --- USE TAYLOR SERIES SERIES EXPANSION (INCLUDING SECOND DERIVATIVES)
C        FOUND ON PAGE 362 OF "ADVANCED CALCULUS FOR APPLICATIONS BY
C        HILDEBRAND (FIRST EDITION, FIFTH PRINTING) TO DETERMINE THE ROOTS
C        OF THE FOLLOWING EQUATION, WHICH IS QUADRATIC IN DELTA M.
C
C           F(M0+DELTA M) = 0
C                         = F(M0) + DELTA M * F'(M0)
C                             + (DELTA M)**2 * F''(M0)/2
C
C        HERE M0 IS THE CURRENT ESTIMATE OF M.
C        THE FORMULA YIELDS TWO ROOTS, DELTA MA AND DELTA MB.
C        MA AND MB ARE THE UPDATED VALUES OF M, WHERE
C
C                   M(A,B) = M0 + DELTA M(A,B)
C
C        F(MA) AND F(MB) ARE BOTH EVALUATED.  THE ESTIMATE THAT PRODUCES THE
C        SMALLEST ABSOLUTE VALUE OF F IS CHOSEN FOR THE NEXT ITERATION.
C
C        IF THE QUADRATIC EQUATION DOES NOT HAVE REAL ROOTS, AN
C        APPROXIMATE SOLUTION FOUND ON PAGE 363 OF HILDEBRAND IS USED, I.E.,
C
C             DELTA M = - (F(M0)/F'(M0)) *
C                         (1 + (DELTA M **2) * (F''(M0)/2*F(M0)))
C
C         WHERE ON THE RIGHT-HAND-SIDE OF THE EQN, DELTA M IS TAKEN AS THE
C         FIRST ORDER APPROXIMATION, DELTA M = -F(M0)/F'(M0)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      DO 60 K = 1,NLIM
C
C   --- CALCULATE THE FIRST AND SECOND DERIVATIVES OF THE FUNCTION F
C
        DSUM3 = ZERO
        DDSUM3 = ZERO
        DO 50 I = 1,NT
          DSUM3 = DSUM3+DLOG(ST(I))*(ST(I))**M0*DLOG(ST(I))
          DDSUM3 = DDSUM3 + (DLOG(ST(I)))**3*(ST(I))**M0
  50    CONTINUE
        DSUM2 = SUM3
        DDSUM2 = DSUM3
        DF = (SUM2 * DSUM3 - SUM3 * DSUM2)/(SUM2**2) + ONE/(M0**2)
C
        DDF = ((SUM2 * DDSUM3 - SUM3 * DDSUM2)/SUM2**2)
     $ - (TWO * DSUM2 * (SUM2 * DSUM3 - SUM3 * DSUM2)/SUM2**3)
     $ - TWO/M0**3

        RADICAL = (DF/DDF)**2 - TWO*F/DDF
        IF (RADICAL .GE. ZERO) THEN
C
C   --- CALCULATE THE ROOTS OF THE QUADRATIC EQUATION
C
          RADICAL = DSQRT(RADICAL)
          MA = M0 - (DF/DDF) + RADICAL
          MB = M0 - (DF/DDF) - RADICAL
C
C    --- CALCULATE F(MA), F(MB), AND THE ASSOCIATED SUMS
C
          CALL SUM (MA, SUM2A, SUM3A, FA)
          CALL SUM (MB, SUM2B, SUM3B, FB)
C
C    --- SELECT THE BETTER ROOT BY COMPARING THE ABSOLUTE
C           VALUE OF THE FUNCTION F
C
          IF (DABS(FA) .LE. DABS(FB)) THEN
```

FIG. X1.3 Continued

```
            M0 = MA
            F = FA
            SUM2 = SUM2A
            SUM3 = SUM3A
          ELSE
            M0 = MB
            F = FB
            SUM2 = SUM2B
            SUM3 = SUM3B
          END IF
        ELSE
C
C   --- IF THE ROOTS ARE COMPLEX, USE THE APPROXIMATE SOLUTION
C
          M1 = M0 - (F/DF)*(ONE+F*DDF/(TWO*DF**2))
C
C   --- CALCULATE F(M1) AND ITS ASSOCIATED SUMS
C
          CALL SUM (M1, SUM2, SUM3, F)
          M0 = M1
        END IF
C
C   --- CONVERGENCE CRITERION:
C           COMPARE THE ABSOLUTE VALUE OF THE FUNCTION F
C           WITH A PRESELECTED TOLERANCE
C
        IF (DABS(F) .LE. EPS) GO TO 70
   60 CONTINUE
C
C   --- MAXIMUM NO. OF ITERATIONS REACHED BEFORE SATISFACTORY VALUE OF M FOUND
C
      WRITE(6,100) NLIM
      GO TO 999
C
C   --- SATISFACTORY ESTIMATE OF WEIBULL MODULUS ATTAINED
C
   70 WMT = M0
C
C   --- COMPUTE THE ESTIMATE OF THE WEIBULL CHARACTERISTIC STRENGTH (WCS)
C
      RWMT = 1.0/WMT
      WCS = ((SUM2/NFAIL)**RWMT)*STNORM
      WRITE(6,110) WMT
      WRITE(6,120) WCS
  100 FORMAT(/,2X,'NO SOLUTION FOUND AFTER ',I4,' ITERATIONS OF THE
     $NEWTON-RAPHSON METHOD',/)
  110 FORMAT(/,2X,' THE ESTIMATED WEIBULL MODULUS = ',F8.3,/)
  120 FORMAT(/,2X,' THE ESTIMATED CHARACTERISTIC STRENGTH = ',F8.3,/)
  999 CONTINUE
      STOP
      END

       SUBROUTINE SUM (M, SUM2, SUM3, F)
      IMPLICIT REAL*8 (A-H, O-Z)
      DOUBLE PRECISION ST(1000), M
      COMMON /DATA/ NFAIL, SUM1, NT, ST, ZERO, ONE
      SUM2 = ZERO
      SUM3 = ZERO
      DO 10 I = 1,NT
        SUM2 = SUM2 + ((ST(I))**M)
        SUM3 = SUM3 + (DLOG(ST(I)) * ((ST(I))**M))
   10 CONTINUE
      F = (SUM3/SUM2) - (SUM1/NFAIL) - (ONE/M)
      RETURN
      END
```

FIG. X1.4 Continued

X2. TEST SPECIMENS WITH UNIDENTIFIED FRACTURE ORIGINS

X2.1 Paragraphs 7.3.1.1 to 7.3.1.4 describe four options, (a) through (d), the experimentalist can utilize when unidentified fracture origins are encountered during fractographic examination. The following four subsections further define the four options, and use examples to illustrate appropriate and inappropriate situations for their use.

X2.1.1 Option (a) involves using all available fractographic information to subjectively assign a specimen with an unidentified origin to a previously identified fracture origin classification. Many specimens with unidentified fracture origins have some fractographic information that was judged to be insufficient for positive identification and classification. (It should be noted that the degree of certainty required for "positive identification" of a fracture-initiating flaw varies from one fractographer to another.) In such cases, Option (a) allows the experimentalist the use of the incomplete fractographic information to assign the unidentified fracture origin to a previously identified flaw classification. This option is preferred when partial fractographic information is available. As an example, consider a tensile specimen where fractography was inconclusive. Fractographic markings may have indicated that the origin was located at or very near the specimen surface, but the fracture-initiating flaw could not be positively identified. Other specimens from the sample were positively identified as failing from machining flaws. It is recognized that machining damage is often difficult to discern. Therefore, in this case it would be appropriate to use Option (a) and infer that the origin is machining damage. The test report (see 7.9 and Fig. 3) must clearly indicate each specimen and where this (or any other) option is used for classifying unidentified specimens. The conclusion of machining damage in this example, however, could be erroneous. For instance, the fracture-initiating flaw may have been a "mainstream microstructural feature"[8] **(18)** (which is also typically difficult to resolve and identify) that happen to be located near the specimen surface. The possibility of erroneous classification such as this are unavoidable in the absence of positive identification of fracture origins.

X2.1.2 Option (b) involves assigning the unidentified fracture origin to the fracture origin classification of the test specimen closest in strength. The specimen closest in strength must have a positively identified fracture origin (not one assigned using Options (a) through (d)). As an example of use of this option, consider a tensile specimen that shattered upon failure such that the fracture origin was damaged and lost, but fracture was clearly initiated from an internal flaw. Other specimens from the sample included positive identification of inclusions and large pores as two active volume-distribution fracture origin classifications. When the fracture strengths from the total data set were ordered, the specimen closest in strength to the specimen with the unidentified fracture origin was the specimen that failed from an inclusion. Use of Option (b) for this test specimen would then allow the unidentified origin to be classified as an inclusion. Justification for Option (b) arises from the tendency of concurrent (competing) flaw distributions to group together specimens with the same origin classification when the test specimens are listed in order of fracture strength. Therefore, the most likely fracture origin classification of a random unidentified specimen is the classification of the specimen closest in strength. The above example can be modified slightly to illustrate a situation where Option (b) would be inappropriate. If the fracture origin classification of the specimen closest in strength was a machining flaw, then Option (b) would lead to a conclusion inconsistent with the fractographic observation that failure occurred from an internal flaw. Fractographic evidence should always supersede conclusions from Option (b).

X2.1.3 Option (c) assumes that the unidentified fracture origins belong to a new, unclassified flaw type and treats these fracture origins as a separate flaw distribution in the censored data analysis. This may occur when the fractographer cannot recognize the flaw type because features of the flaw are particularly subtle and difficult to resolve. In such cases, the fractographer may consistently fail to locate and classify the fracture origin. Examples of flaw types that are difficult to identify include: machining damage, zones of atypically high microporosity, and mainstream microstructural features. Option (c) may be appropriate if a set of specimens with unidentified fracture origins have similar and apparently related features. Unfortunately, there are many situations where Option (c) is incorrect and where use of this option could result in substantial errors in parameter estimates. For instance, consider the case where several unidentified specimens are concentrated in the upper tail (high strength) of the strength distribution. These fracture origins may belong to a classification that has been previously identified, but the smaller flaws at the origins were harder to locate, or possibly the origins were lost due to the greater fragmentation associated with high-strength specimens. Use of Option (c) to treat these high-strength specimens as a new flaw classification would create a bias error of unknown magnitude in the parameter estimates of the proper flaw classification.

X2.1.4 Option (d) involves the removal of test specimens with unidentified fracture origins from the sample (that is, the strengths are removed from the list of observed strengths). This option is rarely appropriate, and is not recommended by this practice unless there is clear justification. Option (d) is only valid when test specimens with unidentified fracture origins are randomly distributed through the full range of strengths and flaw classifications. There are few plausible physical processes that create such a random selection. An example where Option (d) is justified is a data set of 50 specimens where the first 10 fractured specimens (in order of testing) were misplaced or destroyed after testing but prior to fractography. The unidentified specimens were therefore created by a process that is random. That is, the 10 strengths are expected to be randomly distributed through the strength distribution of the remaining 40, and the 10 origin classifications are expected to be randomly distributed through the origin types of the remaining 40. (In this example, Option (b) could also be considered.) Option (d) is not appropriate where unidentified fracture origins are a consequence of high-strength test specimens shattering virulently such that the fragment with the origin is lost. This situation occurs with more frequency in the upper tail (high strength) of the strength distribution, and thus the unidentified fracture origins would not occur at random strengths.

X2.2 Paragraphs X2.2.1 to X2.2.6 expand on the proper use and implementation of the four options described in X2.2.1.

X2.2.1 When partial fractographic information is available, Option (a) is preferred and should be used to incorporate the information as completely as possible into the assignment of fracture origin classification. Option (d) should be used only in unusual situations where a random process for creation of unidentified origins can be justified.

X2.2.2 Situations may arise where more than one option will be used within a single data set. For instance, of five

[8] "Mainstream microstructural features" or "ordinary microstructural features" are fracture origins that occur at features such as very large grains that are part of the ordinary distribution of the microstructure, albeit at the large end of the distribution of such features. These are distinguished from abnormal microstructural features such as inclusions or grossly large pores.

specimens with unidentified origins, three might be classified based on partial fractographic information using Option (a), while the remaining two, which have no fractographic hints, might then be classified using Option (b).

X2.2.3 When specimens with unidentified fracture origins are contained within a data set, the test report (see 7.9) must include a full description of which specimens were unidentified, and which option or options were used to classify the specimens.

X2.2.4 If the unidentified fracture origins occur frequently in the lower tail of the strength distribution, then caution and extra attention is warranted. Strength analyses are typically extrapolated to lower strengths and lower probabilities of failure than those observed in the data set. Proper statistical evaluation and assignment of fracture origin classifications near the lower-strength tail is therefore particularly important because the low-strength distribution typically dominates extrapolations of this type.

X2.2.5 When only a few fracture origins are unidentified, effects of incorrect classification are minimal. When more than 5 or 10 % of the origins are unidentified, substantial statistical bias in estimates of parameters can result. When used for design applications, proper choice of options from X2.1 is critical and should be carefully justified in the test report. In such design applications, it may be prudent to carry out the analysis for more than one option to determine the sensitivity to choice of an improper option. For instance, in a group of 50 specimens with 10 unidentified origins (no partial fractographic information), the analysis could be conducted first using Option (b) then again using Option (c). The results from the two analyses could then be used individually to estimate the behavior of the designed component. If a conservative prediction of component behavior is warranted, the more conservative result of the two analyses should be used.

X2.2.6 Finally, if most or all of the test specimens within a sample contain unidentified fracture origins, then censored data analysis according to this practice is not possible. The strengths should be plotted on Weibull probability axes and, if the data reveal a pronounced bend (concave upwards) which is characteristic of two or more concurrent flaw distributions, then the methods described in this practice cannot be used without further refinements.

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